

Decomposing the Complete r -Graph

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Abstract

Let $f_r(n)$ be the minimum number of complete r -partite r -graphs needed to partition the edge set of the complete r -uniform hypergraph on n vertices. Graham and Pollak showed that $f_2(n) = n - 1$. An easy construction shows that $f_r(n) \leq (1 - o(1))\binom{n}{\lfloor r/2 \rfloor}$ and it has been unknown if this upper bound is asymptotically sharp. In this paper we show that $f_r(n) \leq (\frac{14}{15} + o(1))\binom{n}{\lfloor r/2 \rfloor}$ for each even $r \geq 4$.

Keywords: Hypergraph, Decomposition, Graham-Pollak

1 Introduction

The edge set of K_n , the complete graph on n vertices, can be partitioned into $n - 1$ complete bipartite subgraphs: this may be done in many ways, for example by taking $n - 1$ stars centred at different vertices. Graham and Pollak [7, 8] proved that the number $n - 1$ cannot be decreased. Several other proofs of this result have been found, by Tverberg [12], Peck [9], and Vishwanathan [13, 14].

Generalising this to hypergraphs, for $n \geq r \geq 1$, let $f_r(n)$ be the minimum number of complete r -partite r -graphs needed to partition the edge set of $K_n^{(r)}$, the complete r -uniform hypergraph on n vertices (i.e., the collection of all r -sets from an n -set). Thus the Graham-Pollak theorem asserts that $f_2(n) = n - 1$. For $r \geq 3$, an easy upper bound of $\binom{n - \lceil r/2 \rceil}{\lfloor r/2 \rfloor}$ may be obtained by generalising the star example above. Indeed, having ordered the vertices, consider the collection

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of r -sets whose $2nd, 4th, \dots, (2\lfloor r/2 \rfloor)th$ vertices are fixed. This forms a complete r -partite r -graph, and the collection of all $\binom{n - \lfloor r/2 \rfloor}{\lfloor r/2 \rfloor}$ such is a partition of $K_n^{(r)}$. (There are many other constructions achieving the exact same value - see, for example Alon's recursive construction in [3].)

Alon [3] showed that $f_3(n) = n - 2$. More generally, for each fixed $r \geq 1$, he showed that

$$\frac{2}{\binom{2\lfloor r/2 \rfloor}{\lfloor r/2 \rfloor}}(1 + o(1)) \binom{n}{\lfloor r/2 \rfloor} \leq f_r(n) \leq (1 - o(1)) \binom{n}{\lfloor r/2 \rfloor},$$

where the upper bound is from the construction above.

The best known lower bound for $f_r(n)$ was obtained by Cioabă, K ungden and Verstra ete [5], who showed that $f_{2k}(n) \geq \frac{2\binom{n-1}{k}}{\binom{2k}{k}}$. For upper bounds for $f_r(n)$, the above construction is not sharp in general. Cioabă and Tait [6] showed that $f_6(8) = 9 < \binom{8-3}{3}$, and used this to give an improvement in a lower-order term, showing that $f_{2k}(n) \leq \binom{n-k}{k} - 2 \lfloor \frac{n}{16} \rfloor \binom{\lfloor \frac{n}{2} \rfloor - k + 3}{k-3}$ for any $k \geq 3$. (We mention briefly that any improvement of $f_4(n)$ for any n will further improve the above upper bound. Indeed, one can check that $f_4(7) = 9 < \binom{7-2}{2}$, and this will imply that $f_r(n) \leq \binom{n - \lfloor r/2 \rfloor}{\lfloor r/2 \rfloor} - cn^{\lfloor r/2 \rfloor - 1}$ for some positive constant c . But note that, again, this is only an improvement to a lower-order term.)

Despite these improvements, the asymptotic bounds of Alon have not been improved. Perhaps the most interesting question was whether the asymptotic upper bound is the correct estimate.

Our aim of the paper is to show that the asymptotic upper bound is not correct for each even $r \geq 4$. In particular, we will show that

$$f_4(n) \leq \frac{14}{15}(1 + o(1)) \binom{n}{2},$$

and obtain the same improvement of $\frac{14}{15}$ for each even $r \geq 4$.

A key to our approach will be to consider a related question: what is the minimum number of products of complete bipartite graphs, that is, sets of the form $E(K_{a,b}) \times E(K_{c,d})$, needed to partition $E(K_n) \times E(K_n)$? There is an obvious guess, namely that we take the product of the complete bipartite graphs in the partitions of both K_n s. This gives a partition using $(n - 1)^2$ products of complete bipartite graphs. But can we improve this? Writing $g(n)$ for the minimum value, it will turn out that, unlike for f_4 , any improvement in the value of $g(n)$ for one n gives an asymptotic improvement for g as well. In this sense, this means that g is a 'better' function to investigate than f_4 .

The plan of the paper is as follows. In Section 2, we show how the function g is related to f_4 , and give some related discussions. Then in Section 3, we investigate the simplest product of complete graphs: we attempt to partition the product set $E(K_3) \times E(K_n)$ into products of complete bipartite graphs. Although Section 3 is not strictly needed for our final bounds, it does provide several ideas and motivation for later. In Section 4, we prove our main result on g and from this on f_4 . Finally, in Section 5, we mention some remarks and open problems.

We use standard graph and hypergraph language throughout the paper. For an r -uniform hypergraph H , let $f_r(H)$ be the minimum number of complete r -partite r -uniform hypergraphs needed to partition the edge set of H . So $f_r(K_n^{(r)})$ is just $f_r(n)$. A *minimal decomposition* of an r -graph H is a partition of the edge set of H into $f_r(H)$ complete r -partite r -graphs. A *block* is a product of the edge sets of two complete bipartite graphs. For graphs G and H , let $g(G, H)$ be the minimum number of blocks needed to partition the set $E(G) \times E(H)$. Thus $g(n) = g(K_n, K_n)$. Similarly, a *minimal decomposition* of $E(G) \times E(H)$ is a partition of the set into $g(G, H)$ blocks.

2 Products of complete bipartite graphs

We start by showing how g is related to f_4 .

Proposition 1. *Let $\alpha > 0$ be a constant. If $g(n) \leq \alpha n^2$ for all n , then $f_4(n) \leq \alpha(1 + o(1))\frac{n^2}{2}$.*

Proof. We will show that

$$f_4(n) \leq \alpha \left(\frac{n^2}{2} \right) + Cn \log n \quad (1)$$

for some sufficiently large C . This is clearly true for $n \leq 4$. So assume $n > 4$ and the inequality (1) holds for $1, 2, \dots, n-1$. We will consider the case when n is even - the case when n is odd is similar.

In order to decompose the edge set of $K_n^{(4)}$, we can split the n vertices into two equal parts, say $V(K_n^{(4)}) = A \cup B$, where $|A| = |B| = n/2$. The sets of 4-edges $\{e : e \subset A\}$ and $\{e : e \subset B\}$ can each be decomposed into $f_4(n/2)$ complete 4-partite 4-graphs; the sets of 4-edges $\{e : |e \cap A| = 3\}$ and $\{e : |e \cap B| = 3\}$ can each be decomposed into $f_3(n/2)$ complete 4-partite 4-graphs; while the remaining set of 4-edges $\{e : |e \cap A| = |e \cap B| = 2\}$ can be decomposed into $g(n/2)$ complete 4-partite 4-graphs. So by the assumption of $g(n)$ and the induction hypothesis, we have

$$\begin{aligned} f_4(n) &\leq 2f_4(n/2) + g(n/2) + 2f_3(n/2) \\ &\leq 2 \left(\alpha \left(\frac{n^2}{8} \right) + \frac{Cn}{2} \log \left(\frac{n}{2} \right) \right) + \alpha \left(\frac{n}{2} \right)^2 + 2 \left(\frac{n}{2} - 2 \right) \\ &\leq \alpha \left(\frac{n^2}{2} \right) + Cn \log n. \end{aligned}$$

□

In the Introduction, we mentioned that any improvement in the upper bound of $f_4(n)$ from the easy upper bound of $\binom{n-2}{2}$, for one fixed n , will lead to an improvement for all (greater) values of n , but not an asymptotic improvement. However, very helpfully, this is not the case for g . Indeed, any improvement to $g(n)$ for one particular n leads to an asymptotic improvement. This is the content of the following simple proposition.

Proposition 2. Suppose $g(K_a, K_b) < (a-1)(b-1)$ for some a and b . Then $g(n) \leq \beta n^2$ for all n , for some constant $\beta < 1$.

Proof. Suppose $g(K_a, K_b) = c < (a-1)(b-1)$ for some fixed a and b . Then, setting $\alpha = \frac{c}{(a-1)(b-1)}$, we will show that

$$g(K_{1+(a-1)i}, K_{1+(b-1)j}) \leq \alpha((a-1)i)((b-1)j) = cij$$

for any $i, j \geq 1$. This will then imply that $g(1 + (a-1)(b-1)k) \leq \alpha((a-1)(b-1)k)^2$ for any $k \geq 1$, and hence

$$g(n) \leq \alpha n^2 + Cn$$

for some constant C .

We proceed by induction on i . We will show the base case of $g(K_a, K_{1+(b-1)j}) \leq cj$ by induction on j . The case $j = 1$ is true by assumption. So fix $j > 1$ and by induction, we have $g(K_a, K_{1+(b-1)(j-1)}) \leq c(j-1)$.

Let $G = K_b$ be a subgraph of $K_{1+(b-1)j}$. Note that $K_{1+(b-1)j} - G$ (i.e., the graph $K_{1+(b-1)j}$ with the edges of G removed) is a blow-up of $K_{1+(b-1)(j-1)}$ by replacing one of its vertices with an empty graph on b vertices. So $g(K_a, K_{1+(b-1)j} - G) = g(K_a, K_{1+(b-1)(j-1)}) \leq c(j-1)$, implying

$$\begin{aligned} g(K_a, K_{1+(b-1)j}) &\leq g(K_a, G) + g(K_a, (K_{1+(b-1)j} - G)) \\ &\leq g(K_a, K_b) + c(j-1) \\ &\leq cj. \end{aligned}$$

Now fix $i > 1$ and assume the theorem is true for $i-1$. That is,

$$g(K_{1+(a-1)(i-1)}, K_{1+(b-1)j}) \leq c(i-1)j$$

for all $j \geq 1$. To decompose $E(K_{1+(a-1)i}) \times E(K_{1+(b-1)j})$ for any fixed j , we first let $H = K_a$ and note that $K_{1+(a-1)i} - H$ is a blow-up of $K_{1+(a-1)(i-1)}$ by replacing one of its vertices with an empty graph on a vertices. Therefore,

$$\begin{aligned} g(K_{1+(a-1)i}, K_{1+(b-1)j}) &\leq g(H, K_{1+(b-1)j}) + g((K_{1+(a-1)i} - H), K_{1+(b-1)j}) \\ &\leq g(K_a, K_{1+(b-1)j}) + g(K_{1+(a-1)(i-1)}, K_{1+(b-1)j}) \\ &\leq cj + c(i-1)j \quad (\text{by the base case and induction hypothesis}) \\ &= cij. \end{aligned}$$

This completes the proof of the proposition. \square

From Proposition 1 and Proposition 2, in order to improve the asymptotic upper bound on $f_4(n)$, it is enough to find a and b such that $g(K_a, K_b) < (a-1)(b-1)$.

The rest of this section is a digression (and so could be omitted if the reader wishes). The question of whether or not $g(n) = (n-1)^2$ has the flavour of a ‘product’ question. Indeed, it is

an example of the following general question. Suppose we have a set X and a family \mathcal{F} of some subsets of X , and we write $c(X, \mathcal{F})$ for the minimum number of sets in \mathcal{F} needed to partition X . Is it true that $c(X \times Y, \mathcal{F} \times \mathcal{G}) = c(X, \mathcal{F})c(Y, \mathcal{G})$, where $\mathcal{F} \times \mathcal{G} = \{F \times G : F \in \mathcal{F}, G \in \mathcal{G}\}$?

This is certainly not *always* true. Indeed, for a simple example, let $X = \{1, 2, \dots, 7\}$ and $\mathcal{F} = \{A \subset X : |A| = 1 \text{ or } 4\}$. Clearly, $c(X, \mathcal{F}) = 4$. But $X \times X$ can be partitioned into four 3 by 4 rectangles and a single point, giving $c(X \times X, \mathcal{F} \times \mathcal{F}) \leq 13$.

However, there are a few cases where such a product theorem is known. For example, Alon, Bohman, Holzman, and Kleitman [4] proved that if X is a finite set of size at least 2, then any partition of X^n into proper boxes must consist of at least 2^n boxes. Here, a *box* is a subset of X^n of the form $B_1 \times B_2 \times \dots \times B_n$, where each B_i is a subset of X . A box is *proper* if B_i is a proper subset of X for every i . Note that this corresponds to a product theorem where \mathcal{F} is the family of all proper subsets of X . (There are also some related results by Ahlswede and Cai in [1, 2].)

Unfortunately, we have not been able to prove any product theorem that might relate to our problem about $g(n)$. Indeed, it seems difficult to extend the result of Alon, Bohman, Holzman, and Kleitman at all. For example, here are two closely related problems that we cannot solve.

A box is *odd* if its size is odd. Let X be a finite set such that $|X|$ is odd. We can partition X^n into 3^n odd proper boxes - can we do better?

Question 3. *Let X be a finite set such that $|X|$ is odd. Must any partition of X^n into odd proper boxes consist of at least 3^n boxes?*

We do not even see how to answer this question when $|X| = 5$.

A collection of proper boxes $B^{(1)}, B^{(2)}, \dots, B^{(m)}$ of X^n is said to form a *uniform cover* of X^n if every point of X^n is covered the same number of times.

Question 4. *Let X be such that $|X| \geq 2$. Suppose $B^{(1)}, B^{(2)}, \dots, B^{(m)}$ forms a uniform cover of X^n . Must we have $m \geq 2^n$?*

3 Decomposing $E(K_3) \times E(K_n)$

In this section, we investigate $g(K_3, K_n)$. As we know, we can decompose $E(K_3) \times E(K_n)$ using $2(n-1)$ blocks, and the question is whether we can improve this.

It turns out that the Graham-Pollak theorem actually gives some restriction on how small $g(K_3, K_n)$ can be. To be more precise, we will need a weighted version of the Graham-Pollak theorem. For the sake of completeness, we will include a proof here, although we stress that this is just a rewriting of the usual proof of the Graham-Pollak theorem.

Given a graph G and a real number α , we write $\alpha \cdot G$ for the weighted graph where each edge of G is given a weight of α . A collection of subgraphs G_1, G_2, \dots, G_m of K_n is a *weighted*

decomposition of K_n if there exists real numbers $\alpha_1, \alpha_2, \dots, \alpha_m$ such that for each edge e of K_n we have $\sum_{i:e \in G_i} \alpha_i = 1$. Note that the α_i are allowed to be negative.

Theorem 5. *The minimum number of complete bipartite graphs needed to form a weighted decomposition of K_n is $n - 1$.*

Proof. Let the vertex set of K_n be $V = \{1, 2, \dots, n\}$ and associate each vertex i with a real variable x_i . Let G be a complete bipartite subgraph of K_n with vertex classes X and Y . Then we can define $Q(G) = L(X) \cdot L(Y)$, where $L(A) = \sum_{i \in A} x_i$ for any subset $A \subset V$.

Suppose the bipartite graphs $G_k, 1 \leq k \leq q$ with vertex classes X_k and Y_k form a weighted decomposition of K_n . Then we must have

$$\sum_{i < j} x_i x_j = \sum_{k=1}^q \alpha_k L(X_k) L(Y_k)$$

for some real $\alpha_1, \alpha_2, \dots, \alpha_q$. Rewriting the left-hand-side of the above equation, we have

$$\left(\sum_{i=1}^n x_i \right)^2 - \sum_{i=1}^n x_i^2 = 2 \sum_{k=1}^q \alpha_k L(X_k) L(Y_k).$$

It follows that the linear subspace of \mathbb{R}^n determined by the $q + 1$ linear equations $\sum_{i=1}^n x_i = 0$ and $L(X_i) = 0, 1 \leq i \leq q$, must be the zero subspace. Hence $q + 1 \geq n$. \square

Proposition 6. *For $n \geq 2$ we have*

$$\frac{9}{5}(n-1) \leq g(K_3, K_n) \leq 2(n-1).$$

Proof. The upper bound has been explained already. For the lower bound, suppose the blocks H_1, H_2, \dots, H_q form a decomposition of $E(K_3) \times E(K_n)$. Then for each edge $e \in E(K_n)$, restricting the decomposition to the subset $E(K_3) \times e$, one of the following happens: either the three elements of $E(K_3) \times e$ decompose into three different H_i , or else two of the sets are in the same H_i for some i and the third set is in H_j for some $j \neq i$.

Let G_0 be the subgraph of K_n spanned by the set of e such that the first of these happens, and G_1, G_2, G_3 be the subgraphs of K_n spanned by the set of e for each of the three possible ways for the second case to happen, respectively. Thus in total we have

$$q \geq f_2(G_1) + f_2(G_2) + f_2(G_3) + f_2(G_0 \cup G_1) + f_2(G_0 \cup G_2) + f_2(G_0 \cup G_3). \quad (2)$$

Now, since G_0, G_1, G_2, G_3 form a partition of the edge set of K_n , we must have

$$f_2(G_i) + f_2(G_j) + f_2(G_0 \cup G_k) \geq n - 1 \quad (3)$$

for any $\{i, j, k\} = \{1, 2, 3\}$. Next, note that $1 \cdot (G_0 \cup G_i), 1 \cdot (G_0 \cup G_j), (-1) \cdot (G_0 \cup G_k), 2 \cdot G_k$ form a weighted decomposition of K_n for any $\{i, j, k\} = \{1, 2, 3\}$, so by Theorem 5, we must have

$$f_2(G_0 \cup G_1) + f_2(G_0 \cup G_2) + f_2(G_0 \cup G_3) + f_2(G_i) \geq n - 1 \quad (4)$$

for any $i = 1, 2, 3$.

Let $x = \frac{1}{3}(f_2(G_1) + f_2(G_2) + f_2(G_3))$ and $y = \frac{1}{3}(f_2(G_0 \cup G_1) + f_2(G_0 \cup G_2) + f_2(G_0 \cup G_3))$. Summing over different $\{i, j, k\}$ for inequality (3), we get $2x + y \geq n - 1$; while summing over different i for inequality (4), we get $x + 3y \geq n - 1$. This implies that $x + y \geq \frac{3}{5}(n - 1)$, and together with inequality (2), we conclude that

$$\begin{aligned} q &\geq 3x + 3y, \\ \text{i.e. } q &\geq \frac{9}{5}(n - 1). \end{aligned}$$

□

Note that for any partition of K_n into G_0, G_1, G_2, G_3 , we do obtain that $g(K_3, K_n)$ is at most the right-hand-side of (2).

We believe that the only restriction on $g(K_3, K_n)$ should be the restriction coming from the Graham-Pollak theorem, namely that $g(K_3, K_n) \geq \frac{9}{5}(n - 1)$. However, we have been unable to find any decomposition of $E(K_3) \times E(K_n)$ into fewer than $2(n - 1)$ blocks.

Question 7. *Does there exist a constant $\alpha < 2$ such that $g(K_3, K_n) \leq (\alpha + o(1))n$? In particular, can we take $\alpha = \frac{9}{5}$?*

4 Decomposing $E(K_4) \times E(K_n)$

The aim of this section is to find some a, b in which $E(K_a) \times E(K_b)$ can be partitioned into fewer than $(a - 1)(b - 1)$ blocks. In the previous section, we looked at decompositions of $E(K_3) \times E(K_n)$ by considering all the four possible ways to decompose $E(K_3)$ into complete bipartite graphs - this induced four subgraphs that partitioned the edge set of K_n .

Now, those decompositions of K_3 involved three ‘large’ complete bipartite subgraphs (namely, the $K_{1,2}$ s), which between them form a 2-cover of K_3 (each edge of K_3 is in exactly two of them). However, this is in a sense ‘wasteful’, as by the Graham-Pollak theorem, we might expect to find a uniform cover by three ‘large’ complete bipartite subgraphs of K_4 , rather than K_3 .

This suggests that we should look at $E(K_4) \times E(K_n)$ instead of $E(K_3) \times E(K_n)$. It also suggests that, in each $E(K_4) \times e$, we do not allow *any* decomposition of K_4 , but just four decompositions of K_4 , three of which involves a ‘large’ complete bipartite subgraph and the fourth of which consists of single edges. More precisely, the three decompositions of K_4 we allow here each consists of a 4-cycle and two independent edges. The three pairs of independent edges from

these decompositions in turn form another decomposition of K_4 (into six complete bipartite graphs, each of which is a single edge).

Let C_1, C_2, C_3 be the three different 4-cycles of K_4 and let G_0, G_1, G_2, G_3 be the subgraphs of K_n (as in Proposition 6) whose edge sets partition the edge set of K_n . Then the sets $E(C_1) \times E(G_1), E(C_2) \times E(G_2), E(C_3) \times E(G_3), E(K_4 - C_1) \times E(G_0 \cup G_1), E(K_4 - C_2) \times E(G_0 \cup G_2), E(K_4 - C_3) \times E(G_0 \cup G_3)$ form a partition of $E(K_4) \times E(K_n)$. So $E(K_4) \times E(K_n)$ can be decomposed into

$$f_2(G_1) + f_2(G_2) + f_2(G_3) + 2f_2(G_0 \cup G_1) + 2f_2(G_0 \cup G_2) + 2f_2(G_0 \cup G_3)$$

blocks.

By the same argument as in Proposition 6, we have the following.

Proposition 8. *For $n \geq 2$, we have*

$$\frac{12}{5}(n-1) \leq g(K_4, K_n) \leq 3(n-1).$$

Again, it seems plausible that the only constraint on $g(K_4, K_n)$ is the one coming from the Graham-Pollak theorem.

Conjecture 9. $g(K_4, K_n) = \frac{12}{5}(1 + o(1))n$.

While we are unable to resolve this conjecture, we *are* able to find an example with $g(K_4, K_n) < 3(n-1)$. We start by observing that $G_0 \cup G_1, G_0 \cup G_2, G_0 \cup G_3$ form an *odd cover* of K_n (each edge of K_n appears an odd number of times). Now, it is known (see, e.g., [10]) that K_8 has an odd cover with four complete bipartite graphs. Indeed, the four $K_{3,3}$ s with vertex classes $V_1 = \{1, 3, 5\} \cup \{2, 4, 6\}$, $V_2 = \{1, 4, 7\} \cup \{2, 3, 8\}$, $V_3 = \{2, 5, 7\} \cup \{1, 6, 8\}$ and $V_4 = \{3, 6, 7\} \cup \{4, 5, 8\}$ respectively form an odd cover of K_8 . If we break the symmetry by deleting two vertices (vertices 6 and 8) from this odd cover of K_8 , we obtain an odd cover of K_6 by four complete bipartite graphs, two of which are now disjoint. The union of these two disjoint complete bipartite graphs, together with the other two complete bipartite graphs, will be our $G_0 \cup G_1, G_0 \cup G_2, G_0 \cup G_3$. Remarkably, this does give rise to a decomposition of $E(K_4) \times E(K_6)$ into fewer than 15 blocks.

Proposition 10. *The set $E(K_4) \times E(K_6)$ can be decomposed into 14 blocks. In other words, $g(K_4, K_6) \leq 14 < (4-1)(6-1)$.*

Proof. Let G_0, G_1, G_2, G_3 be graphs that form a decomposition of K_6 , defined as follow:

$$\begin{aligned} E(G_0) &= \{12, 34\}, \\ E(G_0 \cup G_1) &= \{ij : i \in \{1, 3, 5\}, j \in \{2, 4\}\}, \\ E(G_0 \cup G_2) &= \{ij : i \in \{1, 4, 6\}, j \in \{2, 3\}\}, \\ E(G_0 \cup G_3) &= \{ij : i \in \{3, 6\}, j \in \{4, 5\}\} \cup \{12, 15, 16\}. \end{aligned}$$

By construction, we have $f_2(G_0 \cup G_1) = f_2(G_0 \cup G_2) = 1$, and $f_2(G_0 \cup G_3) = 2$, and a quick check shows that $f_2(G_1) = f_2(G_2) = f_2(G_3) = 2$. So from the discussion above we have

$$g(K_4, K_6) \leq \sum_{i=1}^3 (f_2(G_i) + 2f_2(G_0 \cup G_i)) = 14.$$

□

Combining Proposition 1, Proposition 2 and Proposition 10, we obtain our main result.

Theorem 11. $f_4(n) \leq \frac{14}{15}(1 + o(1))\binom{n}{2}$.

5 Remarks and open problems

Proposition 10 (together with Proposition 2) implies that $g(n) \leq \frac{14}{15}(1 + o(1))n^2$. We do not believe $\frac{14}{15}$ is the correct constant, but we are not able to improve it. What about a lower bound of $g(n)$? From Proposition 1, we know that if $g(n) = \alpha n^2(1 + o(1))$, then we have $f_4(n) \leq \alpha(1 + o(1))\binom{n}{2}$. So we must have $\alpha \geq \frac{1}{3}$ from Alon's result on the lower bound of $f_4(n)$.

Here, we are able to give a small improvement, namely $\alpha \geq \frac{1}{2}$. For this, we will need a result by Reznick, Tiwari, and West [11] on decomposing weak product graphs into bipartite graphs. The *weak product* $G * H$ of two graphs G and H has vertex set $\{(u, v) : u \in V(G), v \in V(H)\}$ with $(u_1, v_1) \sim (u_2, v_2)$ if and only if $u_1 \sim u_2$ in G and $v_1 \sim v_2$ in H .

Theorem 12 ([11]). *The minimum number of complete bipartite graphs needed to partition the edge set of $K_n * K_n$ is $(n - 1)^2 + 1$.*

Proposition 13. *For $n \geq 2$, we have $g(n) \geq \left\lceil \frac{(n-1)^2 + 1}{2} \right\rceil$.*

Proof. Suppose we can decompose $E(K_n) \times E(K_n)$ into q blocks. For each of such blocks (say the parts from the left K_n are X_1, X_2 and the parts from the right K_n are Y_1, Y_2), we construct two complete bipartite graphs G_1 and G_2 as follows. The vertex classes of G_1 are $\{(x, y) : x \in X_1, y \in Y_1\}$ and $\{(x, y) : x \in X_2, y \in Y_2\}$; while the vertex classes of G_2 are $\{(x, y) : x \in X_1, y \in Y_2\}$ and $\{(x, y) : x \in X_2, y \in Y_1\}$.

Observe that these $2q$ complete bipartite graphs partition the edge set of the weak product $K_n * K_n$. So by Theorem 12, we must have

$$q \geq \left\lceil \frac{(n-1)^2 + 1}{2} \right\rceil.$$

□

In general, for any fixed k , can we improve the upper bound of $(k-1)(n-1)$ on $g(K_k, K_n)$ in a manner similar to what we have considered for $k=3$ and $k=4$? It seems that perhaps there is no K_k having a ‘better’ allowed sets of decompositions than the four allowed decompositions of K_4 that we used in Section 4. If this is correct, perhaps $\frac{4}{5}$ is the right constant even for $g(n)$.

Question 14. *Is it true that $g(n) = \frac{4}{5}(1 + o(1))n^2$?*

Finally, let us turn our attention to the function f_r for $r > 4$. For fixed $r \geq 1$, let α_r be the smallest α such that $f_r(n) \leq \alpha(1 + o(1))\binom{n}{\lfloor r/2 \rfloor}$. Thus the initial construction gives $\alpha_r \leq 1$ for all r , while Theorem 11 says that $\alpha_4 \leq \frac{14}{15}$. This implies that $\alpha_r \leq \frac{14}{15}$ for all even r .

Theorem 15. *For each fixed $k \geq 2$, we have*

$$f_{2k}(n) \leq \frac{14}{15}(1 + o(1))\binom{n}{k}.$$

Proof. We use induction on k . By Theorem 11, the result is true for the base case $k=2$. For larger k , the result is an easy consequence of the following inequality:

$$f_{2k+2}(n) \leq f_{2k}(n-2) + f_{2k}(n-3) + \dots + f_{2k}(2k).$$

This inequality is obtained by ordering the n vertices and observing that the set of $(2k+2)$ -edges whose second vertex is i , for any fixed $i \in \{2, 3, \dots, n-2k\}$, may be decomposed into $f_{2k}(n-i)$ complete $(2k+2)$ -partite $(2k+2)$ -graphs. \square

We do not see how to obtain a bound below 1 for α_r for r odd. But actually we would expect the following to be true.

Conjecture 16. *We have $\alpha_r \rightarrow 0$ as $r \rightarrow \infty$.*

To prove this, it would be sufficient to show that $\alpha_5 < 1$. Indeed, suppose $f_5(n) \leq (\alpha + o(1))\binom{n}{2}$ for some $\alpha < 1$. Let $r = 6k-1$ and order the n vertices. We can decompose the complete r -graph on n vertices by considering the set of r -edges whose 6th, 12th, \dots , $6(k-1)$ th are i_1, i_2, \dots, i_{k-1} respectively, where $i_1 \geq 6$ and $i_{k-1} \leq n-5$ and $i_j - i_{j-1} \geq 6$ for $2 \leq j \leq k-1$. For each such fixed i_1, i_2, \dots, i_{k-1} , these r -edges can be decomposed into $f_5(i_1-1)f_5(i_2-i_1-1) \dots f_5(i_{k-1}-i_{k-2}-1)f_5(n-i_{k-1})$ complete r -partite r -graphs. Summing over all possible choices of i_1, i_2, \dots, i_{k-1} , we deduce that $f_{6k-1}(n) \leq (\alpha^k + o(1))\binom{n}{3k-1}$.

Annoyingly, we do not see how to use any of our arguments about f_4 for f_5 .

Question 17. *Is $\alpha_5 < 1$? In other words, do we have $f_5(n) \leq (\alpha + o(1))\binom{n}{2}$ for some $\alpha < 1$?*

References

- [1] R. Ahlswede and N. Cai, *On extremal set partitions in Cartesian product spaces*, Combinatorics, Probability and Computing **2** (1993), 211–220.
- [2] R. Ahlswede and N. Cai, *On partitioning and packing products with rectangles*, Combinatorics, Probability and Computing **3** (1994), 429–434.
- [3] N. Alon, *Decomposition of the complete r -graph into complete r -partite r -graphs*, Graphs and Combinatorics **2** (1986), 95–100.
- [4] N. Alon, T. Bohman, R. Holzman and D.J. Kleitman, *On partitions of discrete boxes*, Discrete Mathematics **257** (2002) 255–258.
- [5] S.M. Cioabă, A. Küngden and J. Verstraëte, *On decompositions of complete hypergraphs*, Journal of Combinatorial Theory Series A **116** (2009) 1232–1234.
- [6] S.M. Cioabă and M. Tait, *Variations on a theme of Graham and Pollak*, Discrete Mathematics **313** (2013) 665–676.
- [7] R.L. Graham and H.O. Pollak, *On the addressing problem for loop switching*, Bell System Technical Journal **50**(8) (1971) 2495–2519.
- [8] R.L. Graham and H.O. Pollak, *On embedding graphs in squashed cubes*, in: Graph Theory and Applications, in: Lecture Notes in Mathematics, vol. 303, Springer, 1972, pp. 99–110.
- [9] G. Peck, *A new proof of a theorem of Graham and Pollak*, Discrete Mathematics **49** (1984) 327–328.
- [10] J. Radhakrishnan, P. Sen and S. Vishwanathan, *Depth-3 arithmetic circuits for $S_n^2(X)$ and extensions of the Graham-Pollack Theorem*, in: FST TCS 2000: Foundations of Software Technology and Theoretical Computer Science, New Delhi, in: Lecture Notes in Computer Science, Springer, 2000, pp. 176–187.
- [11] B. Reznick, P. Tiwari and D.B. West, *Decomposition of product graphs into complete bipartite subgraphs*, Discrete Mathematics **57** (1985) 189–193.
- [12] H. Tverberg, *On the decomposition of K_n into complete bipartite graphs*, Journal of Graph Theory **6** (1982) 493–494.
- [13] S. Vishwanathan, *A polynomial space proof of the Graham-Pollak theorem*, Journal of Combinatorial Theory, Series A **115** (2008) 674–676.
- [14] S. Vishwanathan, *A counting proof of the Graham-Pollak Theorem*, Discrete Mathematics **313**(6) (2013) 765–766.